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Exam Fluid Dynamics

June 21st, 2019: 14.00-17.00.

This exam has 6 questions. The maximum number of points you can earn by answering the questions is 36. The final exam grade is 1 + (number of earned)/4. Write on each page your name and student number. The use of annotations, books and calculators is not permitted in this examination. All answers must be supported by work. Good luck!.

1. A fluid moves two-dimensionally so that its velocity is given by $(u, v) = (y, -x)$.
   (a) (2 pt) Compute the Lagrangian coordinates of a fluid particle initially at (a, b).
   (b) (1 pt) Show that the particle paths are circles, and descibe the motion in words.

2. (4 pt) The volume flux F of incompressible fluid crossing a closed surface S is defined as
$$F = \int_S u \cdot n \, dS,$$
where u is the fluid velocity and n is the normal to the surface S. By considering the volume flux though the surface of an arbitrary subdomain of the fluid domain D, show that if there are no sources or sinks of fluid within D then u satisfies
$$\operatorname{div} u = 0$$
within D.

3. We consider a non-viscous, incompressible fluid. The velocity field is given by $v(x, t)$. The mass density ρ is constant (since the fluid is incompressible); here we take $\rho = 1$. The pressure is denoted by $p(x, t)$. The pressure is the only surface force. There are no external forces.
   (a) (3 pt) Show that the acceleration of a particle that moves with the fluid flow is given by
$$\frac{\partial v}{\partial t} + (v \cdot \nabla) v$$
   (b) (2 pt) Derive the Euler equations from Newton's second law.
   (c) The vorticity is given by $\omega = \nabla \times v$.
   (i) (2 pt) Show, using (b), that
$$\frac{\partial \omega}{\partial t} + (v \cdot \nabla) \omega = (\omega \cdot \nabla) v$$
   Here the identity $(v \cdot \nabla) v = \omega \times v + \frac{1}{2} \nabla (|v|^2)$ may be used (without proof).
   (ii) (1 pt) Argue that if the vorticity ω vanishes at a certain time $t = t_0$, then $\omega = 0$ for all $t > t_0$ (in the absense of boundary conditions).

(d) (3 pt) An irrotational flow can be described by a velocity potential Φ, i.e. $v = \nabla\Phi$. Show that for irrotational flows we have (Bernoulli's law):

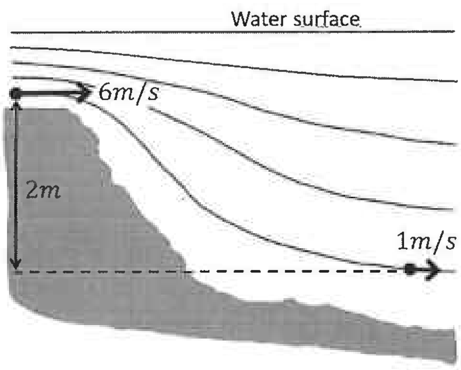
$$\frac{\partial \Phi}{\partial t} + \tfrac{1}{2}|\nabla\Phi|^2 + p = C(t)$$

where $C(t)$ is an arbitrary function of time.

4. Consider steady flow of an incompressible inviscid fluid taking gravity forces into account. The (constant) density is denoted by ρ.

   (a) (2 pt) Formulate Bernoulli's law for this situation. Assume here that the positive y-axis is pointing vertically upward, and the gravity force is directed vertically downward.

   (b) (2 pt) In the figure below, river water (with $\rho = 1000\,\mathrm{kg\,m^{-3}}$) is flowing along the top of a rock. After passing the rock, the water enters a deeper part of the river. A number of streamlines are displayed in the figure. We consider two points on the lowest displayed streamline. Their flow velocities and height difference are given. Calculate how much the pressure in the lowest point exceeds the pressure in the highest point. Take $g = 10\,\mathrm{m\,s^{-2}}$.



5. The complex potential of a 2D incompressible, stationairy flow is given by the following function of a complex variable $z = x + \mathrm{i}y$:

$$w(z) = -U z^{\frac{2}{2-\beta}}$$

where the real numbers U and β satisfy: $U > 0$ and $0 < \beta < 1/2$.

   (a) (2 pt) Determine the complex velocity.

   (b) (2 pt) Sketch the flow by depicting a number of characteristic streamlines.

6. We consider a two-dimensional, steady flow of a viscous, incompressible fluid between the planes $y = 0$ en $y = h$. Both planes do not move. At $x = -L/2$, the pressure is prescribed by $p(-L/2, y) = p_1$ for all $0 < y < h$, whereas at $x = L/2$ the pressure is given by $p(L/2, y) = p_2$ (where again $0 < y < h$). The pressure difference $p_1 - p_2$ causes a flow; there are no (other) external forces.

   (a) (1 pt) This flow is governed by the Navier-Stokes equations; give these equations.

   (b) (3 pt) Assume that the velocity vector is given by $(u(x, y), 0)$. Show with the help of (a) that
   (i) u is independent of x;
   (ii) p is independent of y, and $p'(x) = \text{constant}$.

   (c) (2 pt) Show that (Poiseuille flow)

$$u(y) = \frac{1}{2\mu} \frac{dp}{dx} y(y - h)$$

   where μ is the dynamic viscosity.

   (d) (2 pt) Compute the mass flux Q between the two planes $y = 0$ and $y = h$, and show that ('Ohm's law': voltage = resistance times current):

$$p_1 - p_2 = \frac{12\mu L}{\rho h^3} Q$$

   where ρ denotes the mass density.

   (e) (2 pt) Give Reynolds' number. Explain what it means. Are the above expressions valid for all Reynolds numbers?